Space-time Structures from Critical Values in 2D Quantum Gravity [1]

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Abstract

A model for 2D Quantum Gravity is constructed out of the Virasoro group. To this end the quantization of the abstract Virasoro group is revisited. For the critical values of the conformal anomaly c, some quantum operators ($SL(2,R)$ generators) lose their dynamical content (they are no longer conjugated operators). The notion of space-time itself in 2D gravity then arises as associated with this kinematical $SL(2,R)$ symmetry. An ensemble of different copies of AdS do co-exist in this model with different weights, depending on their curvature (which is proportional to $\hbar^2$) and they are connected by gravity operators. This model suggests that, in general, quantum diffemorphisms should not be imposed as constraints to the theory, except for the classical limit.

1 Introduction

The Virasoro group has been used in previous approaches to 2D quantum gravity, leading to the construction of the action functional of 2D Polyakov induced gravity [1] (gravitational Wess-Zumino-Witten action),

$$S=-\frac{1}{48\pi}(\int d^2x\frac{\partial_+F}{\partial_-F}(\frac{\partial_-^3F}{\partial_-F}-2\frac{(\partial_-^2F)^2}{(\partial_-F)^2}))\quad,\quad ds^2=\partial_-F dx^-dx^+. \tag{1}$$

In [2] a coadjoint orbit method was employed, while a Group Approach to Quantization (GAQ) [3] was the main tool in [4].

These approaches share the use of a particular realization of the Virasoro group as the central extension of $diffS^1$ (i.e. $\widetilde{diffS^1}$). Therefore, the space(-time) sub-manifold

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S^1 appears in these constructions in an explicit way from the very beginning and σ, the parameter of such a space, emerges as a variable external to the group. Another external evolution parameter t, the domain of which is not well understood, then enters the theory constituting with σ the space-time manifold.

In our present framework, the Virasoro group is taken firstly as an abstract group and secondly as the only physical input of the theory. In particular, we do not assume the existence *a priori* of a space-time on which a quantum field theory (of gravitation) is constructed, and this entity must be one of the results of our quantization process of a fundamental symmetry group. In fact, the structure of space-time is one of our main objectives.

The method of quantization that we use is GAQ again, although paying special attention to its quantum aspects. We briefly present the fundamentals of such a technique (for more precise details the reader can refer to [3]).

The chief aim of GAQ is the construction of an unitary and irreducible representation of a Lie group G of physical operators. The operators in this algebra are classified into two classes: those providing a central term under commutation, which are the dynamical ones, and the remainder or the kinematical ones. The former ones appear in conjugated pairs and are the basic variables, while the latter ones generate transformations on the former and are eventually written in terms of them. Therefore the presence of central terms is essential for the existence of dynamics. If they do not appear, then the algebra must be (pseudo-)extended, provided that its (pseudo-)cohomology group allows this [5]. Indeed, the treatment of anomalies in this context [6] advises us to extend maximally the original algebra.

The extended group $\tilde{G}$ has the structure of a principal fibre bundle, with fibre $U(1)$ over G, with connection 1-form Θ, selected as a component of the canonical left-invariant 1-form dual to the fundamental (or vertical) vector Ξ. The characteristic subalgebra of Θ, denoted by $\mathcal{G}_\Theta$, generates the characteristic module of Θ -that is, $Ker\Theta \cap Ker d\Theta$- and coincides with the kernel of the Lie-algebra cocycle. It thus contains the non-dynamical generators.

The *prequantization* is accomplished by the regular representation (over the complex $U(1)$-functions defined on the group). This provides us with two copies of generators: left-invariant ($\tilde{X}^L$) and right-invariant ($\tilde{X}^R$) vector fields. The main point is that these two sets of generators do commute, and this allows us to consider one set as the physical operators (namely the right-invariant ones), and to use the others (left-invariant ones) to find a polarization to reduce the representation, thus arriving to the true *quantization* (obviously, we can interchange the role of the left- and right-invariant vector fields, while maintaining the same physical system).

An important feature of the present approach is that no use of Lagrangians, actions, etc., is required to formulate the physical theory. Those objects are of no primary interest to us. Only the Hilbert space and the action of quantum operators on it have such a fundamental meaning. However, GAQ also provides a generalization of the Cartan approach to Mechanics, which is the natural framework in which to discuss the classical aspects of the theory. In fact, the 1-form Θ is the generalization of the Poincaré-Cartan form and

the trajectories of the (left-invariant) vector fields in $\mathcal{G}_\Theta$ constitute the generalized classical equations of motion. The classical solution manifold $\tilde{G}/(G_\Theta \otimes U(1))$ (phase space) is parametrized by the Noether invariants; that is, the functions $i_{\tilde{X}^R}\Theta$, which are constant along the classical trajectories ($L_{\tilde{X}_j} i_{\tilde{X}^R}\Theta = 0,\ \forall\ \tilde{X}_j\ \in\ \mathcal{G}_\Theta$). The presymplectic form dΘ falls down to the phase space defining a true symplectic form and the corresponding Poisson bracket. The action is obtained from Θ by integrating it along certain trajectories $\sigma(\tau)$ on the group $\tilde{G}$.

We emphasize the fact that physical systems are quantum mechanical so that we shall use the previous classical machinary only after the classical limit to identify the underlying physics. This classical identification is of crucial relevance for an abstract approach to quantum theory such as the present one (see section IV in the first reference of [3]).

Once the fundamentals of our technical background are presented, we refine somewhat more the main question; that is, whether the space-time concept itself emerges from our treatment and, if so, how. The answer is that the space-time variables must be sought as related to the operators inside the characteristic subalgebra $\mathcal{G}_\Theta$.

In a classical approach to field theory, the space-time variables are the integration parameters of certain generators inside the subalgebra $\mathcal{G}_\Theta$ which, as stated above, generate movements in phase-space variables (as space-time translations do). Along the corresponding trajectories, the dynamical parameters in the group gain a dependence in these integration parameters, thus becoming fields over them. In fact, in the process of obtaining the classical action functional out of the field Θ, we can identify the space-time variables, after solving the equations of motion for the generators in $\mathcal{G}_\Theta$, as those appearing explicitly in the integration measure. This construction of the space-time support from the group, can be explicitly shown in the case of Poincaré invariant dynamics for the scalar, electromagnetic and Proca fields. In these cases, we can begin from the corresponding groups (see [7] and references there in), without considering the space-time and reconstruct it after the exact resolution of the motion equations. The kinematical symmetry group proves to be contained in the fields group.

However, the natural way of approaching to the space-time underlying a quantum (field) theory would consist of finding the support for the quantum states of the irreducible Hilbert space of the theory, through the C^*-algebra defined by those states.

2 The Virasoro group

In this section, we present a quick survey of the Virasoro group, our starting point being the algebra:

$$\left[\hat{L}_n, \hat{L}_m\right] = (n-m)\,\hat{L}_{n+m} \quad . \tag{2}$$

As stated in the **Introduction**, we shall consider all the central extensions of this algebra, which will decide the dynamical content of the group parameters. Such extensions are:

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n,-m} \quad , \tag{3}$$

where c is the genuine central extension parameter and c' is the parameter of a family of pseudo-extensions (a redefinition of L_0 causing a non-trivial connection form on the group; see [8]).

The next step is to construct a formal group law from this algebra, and this was indeed done in [8]. The resulting expression for the extended group is [4]:

$$\begin{aligned} l''^m &= l^m + l'^m + ipl'^p l^{m-p} + \frac{(ip)^2}{2!} l'^p l^n l^{m-n-p} + ... + \sum_{n_1+...+n_j=-k} \frac{(ip)^r}{r!} l'^p l^{n_1} ... l^{n_r} + ... \\ \varphi'' &= \varphi' + \varphi + \xi_c(g,g') - \frac{c'}{24}\xi_{cob}(g',g) \qquad (4) \\ & \quad \xi_{cob}(g',g) = l^{0''} - l^{0'} - l^0 \end{aligned}$$

(the explicit expression for $\xi_c\,(g,g')$ is rather involved and thus we refer the reader to [8]).

From this group law, we compute the left- and right-invariant vector fields, $\tilde{X}^L_{l^k}$ and $\tilde{X}^R_{l^k}$, respectively. The corresponding expressions are presented in **Appendix B**. We make explicit here only the non-central part of $\tilde{X}^L_{l^k}$:

$$X^L_{l^k} = \frac{\partial}{\partial l^k} + i\,(m-k)\, l^{m-k} \frac{\partial}{\partial l^m} \quad . \qquad (5)$$

The quantization form is obtained by duality on left fields ($\Theta(\Xi) = 1, \;\; \Theta(\tilde{X}^L_{l^n}) = 0$):

$$\begin{aligned} \Theta &= \frac{i}{24}(cn^2 - c')nl^{-n}dl^n + \\ &+ \sum_{\substack{k=2 \\ n_1+...n_k=-n}} \frac{(-i)^k}{24}[cn_1^2 - c' + cn^2 \sum_{m=2}^{k} \frac{1}{m!}] n_1...n_k l^{n_1}...l^{n_k} dl^n + d\varphi \qquad (6) \end{aligned}$$

Especially important in searching for the space-time notion is the structure of the characteristic subalgebra $\mathcal{G}_\Theta$ of Θ, which coincides with the kernel of the Lie algebra cocycle. Thus, depending on the values of c and c', we find:

$$\begin{aligned} \text{i)}\frac{c'}{c} \neq r^2, r \in Z, \;\Rightarrow\; \mathcal{G}_\Theta &= \langle \tilde{X}^L_{l^0} \rangle \qquad (7) \\ \text{ii)}\frac{c'}{c} = r^2, r \in Z, \;\Rightarrow\; \mathcal{G}_\Theta &= \langle \tilde{X}^L_{l^{-r}}, \tilde{X}^L_{l^o}, \tilde{X}^L_{l^r} \rangle \approx sl^{(r)}(2,R) \;\; . \end{aligned}$$

Since we wish to find a two-dimensional space-time inside the group, we must choose ii). Besides, we are searching for a unitary representation of our algebra. This imposes (see [8, 9] and the next section) $c = c'$ $\;(r = 1)$. We must note, however, that, although we need $c = c'$ for implementing a notion of space-time, the dynamics of our system are as

[4] Throughout the text, summation symbols are explicited only in those cases which present a constraint on the indices. In all other cases, wherever an index appears repeated, summation from $-\infty$ to ∞ is understood.

well defined for other values of c and c' (provided that the theory is unitary) but without a notion of space-time as such. Our first conclusion then is that space-time appears as a critical case and outside this critical value of the conformal anomaly, we would still have a well-defined physical system.

Let us now detail the elements of the Cartan-like geometry associated with the Virasoro group in the critical case $c = c'$, which will constitute the mathematical framework of the physical theory underlying the classical limit.

We write the evolution equations for the l^n parameters under the action of $SL(2,R)$. Under this action, l^n are functions of the $SL(2,R)$ parameters, thus becoming, as we stated in the **Introduction**, fields over the $SL(2,R)$ manifold (parametrized by $\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_{-1}$ which are the parameters associated to the fields $\tilde{X}^L_{l^0}, \tilde{X}^L_{l^1}, \tilde{X}^L_{l^{-1}}$, respectively):$l^n = l^n(\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_{-1})$. The dynamical system can then be written out as:

$$\frac{\partial l^n}{\partial \tilde{\lambda}_0} = (\tilde{X}^L_{l^0})^{l^n} \quad , \quad \frac{\partial l^n}{\partial \tilde{\lambda}_1} = (\tilde{X}^L_{l^1})^{l^n} \quad , \quad \frac{\partial l^n}{\partial \tilde{\lambda}_{-1}} = (\tilde{X}^L_{l^{-1}})^{l^n}, \tag{8}$$

(where the l^n component of the field $\tilde{X}^L_{l^i}$, $(\tilde{X}^L_{l^i})^{l^n}$, proves to be $\tilde{X}^L_{l^i}(l^n)$).

Using the explicit expressions for $(\tilde{X}^L_{l^m})^{l^n}$ from (5), we find:

$$\begin{aligned}
\frac{\partial l^m}{\partial \tilde{\lambda}_0} &= iml^m \quad \text{for} \quad m \neq 0 \quad , \quad \frac{\partial l^0}{\partial \tilde{\lambda}_0} = 1 \\
\frac{\partial l^m}{\partial \tilde{\lambda}_1} &= i(m-1)l^{m-1} \quad \text{for} \quad m \neq 1 \quad , \quad \frac{\partial l^1}{\partial \tilde{\lambda}_1} = 1 \\
\frac{\partial l^m}{\partial \tilde{\lambda}_{-1}} &= i(m+1)l^{m+1} \quad \text{for} \quad m \neq -1 \quad , \quad \frac{\partial l^{-1}}{\partial \tilde{\lambda}_{-1}} = 1 \, .
\end{aligned} \tag{9}$$

The solutions to these equations can be obtained exactly:

$$\begin{aligned}
l_0 &= \lambda_0 \\
l_{-1} &= [\lambda_{-1} + \sum_{s=0}^{\infty}(-1)^s \begin{pmatrix} 1+s \\ 1 \end{pmatrix} (i\lambda_1)^s \mathcal{L}_{-1-s}] e^{-i\lambda_0} \\
l_1 &= [\lambda_1 + \sum_{s=0}^{\infty}(-1)^s \begin{pmatrix} 1+s \\ 1 \end{pmatrix} (-i\lambda_{-1})^s \mathcal{L}_{1+s}] e^{i\lambda_0} \\
l_{-n} &= [\sum_{s=0}^{\infty}\sum_{m=0}^{n-2}[(-i)^{s+m} \begin{pmatrix} n-1 \\ m \end{pmatrix} \begin{pmatrix} s+n-m \\ n-m \end{pmatrix} \lambda_1^s \lambda_{-1}^m \mathcal{L}_{-n+m-s}] + \frac{i}{n}(-i\lambda_{-1})^n] e^{-in\lambda_0} \\
l_n &= [\sum_{s=0}^{\infty}\sum_{m=0}^{n-2}[i^{s+m} \begin{pmatrix} n-1 \\ m \end{pmatrix} \begin{pmatrix} s+n-m \\ n-m \end{pmatrix} \lambda_{-1}^s \lambda_1^m \mathcal{L}_{n-m+s}] - \frac{i}{n}(i\lambda_1)^n] e^{in\lambda_0} \quad , n \geq 2 \, ,
\end{aligned} \tag{10}$$

where $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_1 = \lambda_1 e^{i\lambda_0}$, $\tilde{\lambda}_{-1} = \lambda_{-1}e^{-i\lambda_0}$, and the integration constants $\mathcal{L}_n$ ($|\, n \,| \geq 2$), parametrize the solution manifold $Virasoro/SL(2,R)$.

This symplectic manifold, with symplectic form $d\Theta/SL(2,R)$, can also be parametrized by the basic Noether invariants $L_n \equiv i_{X^L_{l^n}}\Theta$, with $|\, n \,| \geq 2$. The Noether invariants

$L_j \equiv i_{X^L_{l^j}}\Theta$ ($j = 0, \pm 1$) must be written in terms of the basic ones. We shall illustrate this fact to the lowest non-trivial order, at which:

$$\begin{aligned} L_k &= i_{X^L_{l^k}}\Theta = \frac{ic}{12}(k^2-1)kl^{-k} - \frac{c}{24}\sum_{n_1+n_2=-k}[(n_1^2 - 1 + \frac{k^2}{2})n_1 n_2 \\ &- (n_1^2-1)n_1 k + \frac{k^2}{2}(n_1^2+n_2^2+n_1 n_2) - \frac{k^2}{2}]l^{n_1}l^{n_2} + ... \end{aligned} \tag{11}$$

From this, we explicitly see that for the kinematical Noether invariants, $(L_0, L_{\pm 1})$, the linear term vanishes and the only contribution to the cuadratic term comes from the basic Noether invariants [5], as should be the case. For these basic Noether invariants, we find $l^n = (\frac{12}{ic(k^2-1)k}L_{-n} + ...)$, $|n| \geq 2$, and the expressions of the kinematical ones are:

$$\begin{aligned} L_0 &= \frac{6}{c}\sum_{n\neq 0,\pm 1}\frac{1}{(n^2-1)}L_{-n}L_n + ... \\ L_1 &= \frac{6}{c}\sum_{n\neq 0,\pm 1,-2}\frac{1}{n(n+1)}L_{-n}L_{n+1} + ... \\ L_{-1} &= \frac{6}{c}\sum_{n\neq 0,\pm 1,2}\frac{1}{n(n-1)}L_{-n}L_{n-1} + ... \end{aligned} \tag{12}$$

The previous parametrization of the solution manifold with L_n ($\mathcal{L}_n$) or, accordingly, of the Virasoro group with l_n corresponds to a Fourier-like description. A configuration-like description will be achieved by defining the field

$$F(\lambda_{-1}, \lambda_0, \lambda_1) = \sum_n l_n(\lambda_{-1}, \lambda_0, \lambda_1)\ , \tag{13}$$

which parallels the standard Fourier expansion of a field, $\phi(x,t) = \sum_k A_k e^{ikx-k_0 t}$, where the constants $\mathcal{L}_n$ in (10) play the role of the A_k's, and the functions of $\lambda_0, \lambda_1, \lambda_{-1}$ accompanying the $\mathcal{L}_n$'s play the role of the exponentials.

Explicitly, and with some abuse of the language concerning the notation of $l_n(\lambda_{-1}, \lambda_0, \lambda_1)$ and $l_n(\lambda_{-1}, \lambda_1)$,

$$\begin{aligned} & F(\lambda_{-1}, \lambda_0, \lambda_1) = \sum_n l_n(\lambda_{-1}, \lambda_0, \lambda_1) = \\ = \ & \lambda_0 + [\sum_{n>0}\sum_{s=0}^{\infty}\sum_{m=0}^{n-2} i^{s+m}\begin{pmatrix} n-1 \\ m \end{pmatrix}\begin{pmatrix} s+n-m \\ n-m \end{pmatrix}\lambda_{-1}^s\lambda_1^m\mathcal{L}_{n-m+s} - \frac{i}{n}(i\lambda_1)^n]e^{in\lambda_0} + \\ + \ & \sum_{n>0}\sum_{s=0}^{\infty}\sum_{m=0}^{n-2}(-i)^{s+m}\begin{pmatrix} n-1 \\ m \end{pmatrix}\begin{pmatrix} s+n-m \\ n-m \end{pmatrix}\lambda_1^s\lambda_{-1}^m\mathcal{L}_{-n+m-s} + \frac{i}{n}(-i\lambda_{-1})^n]e^{-in\lambda_0} = \\ = \ & \lambda_0 + \sum_{n\neq 0} l_n(\lambda_{-1}, \lambda_1)e^{in\lambda_0} \end{aligned} \tag{14}$$

[5] For L_0, L_1 and L_{-1}, the polynomial on n_1 and n_2 in the quadratic term vanishes whenever l^0, l^1 or l^{-1} appear.

Making the change of variables:

$$\begin{aligned} u &= \frac{1}{2}(\lambda_1 + \lambda_{-1}) \\ v &= \frac{1}{2}(\lambda_1 - \lambda_{-1}) \\ \lambda &= \lambda_0 \ , \end{aligned} \tag{15}$$

we express

$$F(u, v, \lambda) = \lambda + \sum_{n \neq 0} l_n(u, v) e^{in\lambda} \ . \tag{16}$$

As we shall see in the next section, the space-time notion is related to that of homogeneous spaces inside $SL(2, R)$. Both de Sitter and anti-de Sitter are found among these spaces, since dS and AdS groups in two dimensions are isomorphous to $SL(2, R)$. AdS geometry can be constructed from the Killing metric,

$$ds^2 = dv^2 + d\bar{\lambda}^2 - du^2 \quad , \tag{17}$$

by imposing the Casimir constraint:

$$v^2 + \bar{\lambda}^2 - u^2 = R^2 \quad , \tag{18}$$

where $\bar{\lambda}$ is the decompactified λ ($\bar{\lambda} = sin^{-1}\lambda$).
The geometry of dS follows from:

$$ds^2 = dv^2 - d\bar{\lambda}^2 + du^2 \quad , \tag{19}$$

with

$$v^2 - \bar{\lambda}^2 + u^2 = R^2 \quad . \tag{20}$$

We see that they are topologically the same (a one-fold hyperboloid), but AdS has negative constant curvature, $K = -\frac{1}{R^2}$, and compact time, while dS has positive constant curvature, $K = \frac{1}{R^2}$, and compact space. In both cases, Minkowski is recovered within the limit $R^2 \rightarrow \infty$ [6].

3 Quantum representations: a model for the quantum theory of gravity

[6]Constant curvature space-times in two dimensions are not a solution to the Einstein field equations with cosmological constant Λ ($\neq 0$) in a vacuum. In higher dimensions, however, they are, and we find $K \approx \Lambda$ [10, 11]. Thus, in the hope that these results can be extended to higher dimensions in a suitable generalization, we are tempted to interpret K as a cosmological constant.

3.1 Algebraic construction

Let us return to the problem of obtaining a unitary, irreducible representation of the Virasoro group. As stated above, this problem was studied in [12, 13, 9, 8] and we take the results from [8].

Two cases are of interest to us: $\frac{c'}{c} \neq r^2$ and $\frac{c'}{c} = 1$. For both, we can find a full (including the entire characteristic subalgebra) and symplectic (including one of the two coordinates of each dynamical pair) polarization:

$$\begin{aligned} \mathcal{P} &= \left\langle \tilde{X}^L_{l^n \leq 0} \right\rangle \quad \text{for} \quad \frac{c'}{c} \neq r^2 \\ \mathcal{P}^{(1)} &= \left\langle \tilde{X}^L_{l^n \leq 1} \right\rangle \quad \text{for} \quad c = c' \ , \end{aligned} \tag{21}$$

and the corresponding polarization conditions for the wave functions Ψ:

$$\begin{aligned} &\tilde{X}^L_{l^n \leq 0} \ \ \Psi = 0 \qquad \text{if} \qquad c \neq c' \ \ (\tilde{X}^L_{l^n} \in \mathcal{P}) \\ &\tilde{X}^L_{l^n \leq 1} \ \ \Psi = 0 \qquad \text{if} \qquad c = c' \ \ (\tilde{X}^L_{l^n} \in \mathcal{P}^{(1)}) \ . \end{aligned} \tag{22}$$

(Note that we can work with the case $c \neq c'$ only because the latter can be formally recovered from the former by making $c = c'$ at the end of the calculations).

The solutions to these polarization equations build the representation Hilbert space. The Virasoro algebra operators are represented by acting with the right-invariant vector fields on these specific polarized functions.

Redefining the generators:

$$\hat{L}_{n \neq 0} = i\tilde{X}^R_{l^n} \quad , \quad \hat{L}_0 \equiv i\tilde{X}^R_{l^0} \quad , \quad I \equiv i\Xi \ , \tag{23}$$

we recover the usual commutators for the Virasoro Lie algebra (these relationships are more usually expressed in terms of (c, h), where $h = \frac{c-c'}{24}$, but we prefer to maintain the (c, c') parameters in which our analysis is more transparent).

It should be pointed out that in these representations there are no null vectors [8]. This is a crucial point, because the space of polarized functions is not irreducible in general (a difference with the compact semisimple group case). Taking advantage of the absence of null vectors, it is possible to consider the orbit of the enveloping algebra through the vacuum and thus to construct an irreducible subspace $\mathcal{H}_{(c,c')}$:

$$\mathcal{H}_{(c,c')} \equiv \left\langle \hat{L}_{n_j} \ldots \hat{L}_{n_1} \ |0\rangle \right\rangle \qquad n \ \leq \ -1 \quad j = 1, 2, 3, \ldots \ \text{for } c \neq c' \tag{24}$$

$$n \ \leq \ -2 \quad j = 1, 2, 3, \ldots \ \text{for } c = c'. \tag{25}$$

These are the representation spaces we shall work with.

With regard to unitarity and irreducibility, some brief comments are relevant:

- Values of c and c' for unitary representations:
  a) $c \geq 1$, with $\frac{c-c'}{24} \geq 0$.
  b) $0 < c < 1$ with:

$$c = 1 - \frac{6}{m(m+1)} \tag{26}$$

$$\frac{c-c'}{24} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \quad (1 \leq s \leq r \leq m-1\ , \quad \text{and } m, r, s \text{ integers with } m \geq 2) \tag{27}$$

  Pairs (c, c') different from the previous ones, lead to non-unitary representations.

- Values of c and c' for reducible representations:

$$c = 1 - \frac{6}{m(m+1)} \tag{28}$$

$$\frac{c-c'}{24} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \quad (r, s \in Z^+\ ,\ m \in R^+).$$

  For $c > 1$, therefore, we have irreducible representations.

For more details about unitarity and irreducibility see [12, 14, 15, 16, 17, 8]. In particular, in the second reference in [8], it is proven that the reduction for $c < 1$ can be achieved by means of higher-order polarizations.

The representation of our original algebra on a Hilbert space has been accomplished. As noted above, making $c = c'$ (that is $h = 0$), the Virasoro representations with $SL^{(1)}(2, R)$ as the characteristic subalgebra, are recovered. This is the case in which a notion of space-time can be found. Under this condition, there are two kinds of operators acting on our Hilbert space:

- Dynamical operators (*gravity* field operators): $\hat{L}_n\ ,\ \mid n \mid \geq 2$

- Space-time operators: $\hat{L}_n\ ,\ \mid n \mid \leq 1$.

As a preliminary approach to the construction of an explicit model for Quantum Gravity problem and, in order to simplify the mathematical issues related with space-time reconstruction, we are going to focus on the case $c > 1$. This condition, together with $c = c' \Leftrightarrow h = 0$, guarantees unitarity, irreducibility and allows for the notion of space-time. Although there are unitary representations with $c = c'$ and $c \leq 1$ (with $r = s$ and thus parametrized by $m \geq 2$), these representations are reducible and we must resort to higher-order polarizations which lead to a non-commutative structure on the C^*-algebra of the functions in the carrier subspace. This problem, although extremely interesting, is beyond the scope of this work.

To begin the study of implementing of the space-time notion, let us consider the reduction of our unitary irreducible representation of the Virasoro group under its space-time subgroup $SL^{(1)}(2,R)$. From the orbit-through-the-vacuum construction for the representation of the Virasoro group, the $SL^{(1)}(2,R)$ representations (which are unitary and thus infinite-dimensional) are of maximal-weight type (see [18]). As can be seen in detail in the **Appendix A**, on each level of the Virasoro representation (that is, the finite-dimensional space of eigenvectors of L_0 with eigenvalue N) there exist $(D^N - D^{N-1})$ maximal weight vectors of $SL^{(1)}(2,R)$, where D^N is the dimension of the N level, given by the number of partitions of N in which 1 is lacking (for instance, for $N=4$, $(2,2)$ is allowed while $(3,1)$ is not). From each of these maximal-weight vectors, an irreducible representation of $SL^{(1)}(2,R)$ with index N, $R^{(N)}$, and with Casimir $N(N-1)$, is constructed.

The reduction of the original Hilbert space, $\mathcal{H}_{(c,c)}$, is then:

$$\mathcal{H}_{(c,c)} = \bigoplus_N (D^{(N)} - D^{(N-1)}) R^{(N)}. \tag{29}$$

It can be shown that these $SL(2,R)$ irreducible representations are orthogonal with the Virasoro scalar product ($\hat{L}_n = \hat{L}^+_{-n}$, $\langle 0 \mid 0 \rangle = 1$), allowing a standard quantum interpretation of the states. We note that $(D^{(N)} - D^{(N-1)})$, the degeneration of the $R^{(N)}$ representation, increases with N.

We give examples of the $SL^{(1)}(2,R)$ representations with the lowest values for N. To do so, we look for $SL^{(1)}(2,R)$ maximal-weight vectors at level N by considering the most general linear combination of Virasoro states of level N and then simply determining the coefficients for which this vector is annihilated by $\hat{L}_1$ (there are $(D^{(N)} - D^{(N-1)})$ solutions: the kernel of $\hat{L}_1$ restricted to level N). The excited $SL^{(1)}(2,R)$ states are established by applying the operator $\hat{L}_{-1}$ successively on the corresponding vacuum.

For $N=1$, there are no Virasoro states (because $c=c'$), and thus there are no $(N=1) - SL(2,R)$ representations.

For $N=2$, we have only the vector $\hat{L}_{-2} \mid 0\rangle$, [7] which is in fact annihilated by $\hat{L}_1$ (as it should be). The excited states are:

$$\mid N=2,n\rangle = C_{2,n}(\hat{L}_{-1})^n \hat{L}_{-2} \mid 0\rangle, \tag{30}$$

where $C_{2,n}$ is a normalization constant.

For $N=3$, $(D^{(3)} - D^{(2)}) = 1-1 = 0$, and therefore there is no $SL^{(1)}(2,R)$ vacuum.

For $N=4$, the only vacuum and the corresponding excited states are:

$$\mid N=4,n\rangle = C_{4,n}(\hat{L}_{-1})^n(-\frac{3}{5}\hat{L}_{-4} + \hat{L}_{-2}\hat{L}_{-2}) \mid 0\rangle. \tag{31}$$

For $N=5$, as for $N=3$, there is no vacuum.

For $N=6$, we have the following vacua (chosen as orthogonal):

$$\mid N=6, n=0,1\rangle \;\; = \;\; C_{6,0,1}(\frac{-1}{7}\hat{L}_{-6} - \frac{8}{5}\hat{L}_{-4}\hat{L}_{-2} + \hat{L}_{-3}\hat{L}_{-3}) \mid 0\rangle \text{ and} \tag{32}$$

[7] $\mid 0\rangle$ is the Virasoro vacuum.

$$
\begin{aligned}
\mid N=6, n=0,2\rangle \;&=\; C_{6,0,2}\big(\tfrac{1183-6080c}{-18646+20160c}\hat{L}_{-6}+\tfrac{28013+211680c}{74584-80640c}\hat{L}_{-4}\hat{L}_{-2} \\
&+\; \hat{L}_{3}\hat{L}_{3}+\tfrac{10365+22400c}{-18646+20160c}\hat{L}_{-2}\hat{L}_{-2}\hat{L}_{-2}\big)\,|0\rangle.
\end{aligned}
$$

which generate the corresponding representations.

Now, with each irreducible representation of $SL^{(1)}(2,R)$ we associate a space-time geometry as the support of the C^{*}-algebra generated by the corresponding carrier space. This construction can be made in general through the Gelfand-Kolmogorov theory [19, 20].

If we considered the case $c \leq 1$ (see (28) for which the standard Verma module approach leads to the existence of null-vector states or, equivalently in our scheme, when the carrier space of the representation is the solution to a higher-order polarization [8]), we should take into account that the C^{*}-algebra constructed from these wave functions would not be a subalgebra of the space of functions on the group (it would not even be commutative) and the general Gelfand-Naimark theory [21] should be used to recover a geometry, which would prove to be non-commutative. Here, we do not undertake the analysis of this interesting case (although we shall do so in the near future), and consider only the simpler representations in which no higher-order polarizations are required so that no non-commutative geometry emerges. In these particular cases ($c > 1$), the process of finding the support space for each C^{*}-algebra generated by a $SL^{(1)}(2,R)$ irreducible representation is not involved to a great degree. In fact, we have only to realize that from a given $SL^{(1)}(2,R)$ irreducible representation $R^{(N)}$, a basis for the complex functions on the hyperboloid (homogeneous space of $SL(2,R)$) can be obtained from the reduction of the tensorial products of $R^{(N)}$ via the Clebsch-Gordan series. In this way, we recover an AdS space-time, which is the homogeneous space associated with the highest-weight representations of $SL(2,R)$[8] , the ones appearing in the Virasoro reduction (dS is linked to non-highest-weight representations). Thus, for each $SL^{(1)}(2,R)$ representation, we have a space-time and, therefore, we find a collection of space-times which are realized simultaneously in the theory.

3.2 Physical interpretation

Before providing a physical interpretation of this model, let us assign dimensions to the objects appearing in it. A glance at the commutation relations of Virasoro algebra (3), reveals that the integers appearing in it have the same dimension as the generators, dimensions which can be determined if we identify (classically) the parameters of the group as space-time variables (see classical motion equations (9)). Thus, the dimension of generators and integers is $(Length)^{-1}$. From this, we conclude that c has dimensions of $(Length)$ and c' of $(Length)^{-1}$.

It is important to redefine the integers as being intrinsically dimensionless (if not, we cannot make a physical analysis of mathematically well-defined expressions such as

[8] In fact, a particle moving on AdS space-time is a physical system whose quantum space of solutions (Hilbert space) is the same of our $SL^{(1)}(2,R)$ representations, and which has an AdS space as the configuration space [23, 22]. This allows us to identify AdS as the space-time associated with the $SL^{(1)}(2,R)$ representations we have found.

$\frac{c'}{c} = r^2$ in (7), because we do not have a scale to determine whether an integer is large or small). Therefore, we should introduce a $(Length)^{-1}$ dimensional constant a and redefine $n \to \frac{1}{a}n$ in all the expressions in the text (we have not done so from the very beginning in order not to create confusion with the existing literature on the Virasoro algebra).

We have encountered three fundamental distances in our model: c, $\frac{1}{c'}$ and a. In the critical case in which space-time appears, there is a relationship between the distances ($c = c'a^2$, the dimensionally correct version of $c = c'$) and we have only two independent ones (c and a, for instance). One of these is related to the notion of long distance in the space-time model (the *radius*), the other with a short one. From arguments to be presented below, we associate c with the long one and a with the short one, while the role of the Planck constant is played by $\frac{a}{c}$.

The constant $\frac{a}{c}$ can be used to redefine the generators in the theory as is usual in Quantum Mechanics:

$$\hat{H}_n = \frac{a}{c}\hat{L}_n \ . \tag{33}$$

We physically interpret each vector in a $SL^{(1)}(2,R)$ representation of index N as a state of the whole space-time defined by this representation. These states are eigenvectors of the kinematical operator $\hat{H}_0$, which can be interpreted as the energy [9]. Thus, the maximal-weight vector of the representation, $|N,0\rangle$, is the fundamental state of the corresponding space-time, while the action of $\hat{H}_{-1}$ moves space-time to excited states:

$$\begin{aligned} \hat{H}_0((\hat{H}_{-1})^n \mid N,0\rangle) &= \frac{a}{c}\frac{(N+n)}{a}(\hat{H}_{-1})^n \mid N,0\rangle \\ Energy(n) &= \frac{N+n}{c}. \end{aligned} \tag{34}$$

The vacuum of the Virasoro representation, $|\,0\rangle$, is interpreted as the physical vacuum of the (whole) Universe [10] in which we do not even have a space-time (is the trivial representation of $SL^{(1)}(2,R)$). The energy of this vacuum is 0, as it should be, but the reason is by no means trivial: it is just a consequence of c and c' being in the critical value $c = c'$.

We have been using the term space-time in the text, while this is not quite precise, as we have no metric notion yet. The reconstruction from the C^*-algebra does not provide a metric. The only primary metric we can consider in the context of our model is the one induced on the hyperboloid from the Killing metric of $SL^{(1)}(2,R)$, which turns out to be AdS metric (as we said before) due the the presence of highest-weight representations. To implement the constraint which allows us to induce this metric, we have to give the *radius* of the hyperboloid. We search in the model for a distance notion which should be completely characterised by the Virasoro representation (i.e. by c) and by the $SL^{(1)}(2,R)$

[9] In fact, the expression of L_0 in terms of the basic variables L_n ($|\,n\,|\geq 2$) (12) is a generalization of the harmonic oscillator energy and parallels the classical version of the Sugawara construction of the Hamiltonian in Conformal Field Theory [24].

[10] We use the term "Universe" in referring to the entire Virasoro representation (the entire physical system), and "space-time" referring to the geometry related to $SL^{(1)}(2,R)$ representations.

representation (N). A length that fulfills these requirements is the Casimir in terms of $\hat{H}_0, \hat{H}_1, \hat{H}_{-1}$:

$$\frac{1}{R^2} \equiv \hat{H}_0^2 - \frac{1}{2}(\hat{H}_1\hat{H}_{-1} + \hat{H}_{-1}\hat{H}_1) = (\frac{a}{c})^2\frac{N(N-1)}{a^2} = \frac{N(N-1)}{c^2}. \tag{35}$$

We have an AdS metric on the hyperboloid given by:

$$\begin{aligned} ds^2 &= dv^2 + d\bar{\lambda}^2 - du^2 \\ v^2 + \bar{\lambda}^2 - u^2 &= \frac{c^2}{N(N-1)}\ , \end{aligned} \tag{36}$$

where u and v are linear combinations of l^1 and l^{-1}, which make the corresponding momentum generators hermitian. Therefore, we have an AdS space-time support.

Up to now, we have been concerned with the $SL^{(1)}(2,R)$ symmetry, which provides an ensemble of AdS space-times with *radii* $\frac{c}{\sqrt{N(N-1)}}$ associated with each $SL^{(1)}(2,R)$ representation. On them the $\hat{H}_{-1}$ operator acts by creating excited states of these space-times. No relationships among the different $SL(2,R)$ representations have been reported. Let us now consider the $\hat{H}_n$ ($\mid n \mid\geq 2$) gravitational modes. As they do not preserve the $SL^{(1)}(2,R)$ representations, they have the effect of transforming a state of a definite space-time, into a linear combination of states of different space-times. That is, if we start from a state of a space-time of *radius* R, after the action of gravity the state that describes space-time is spread over space-times of different *radii*. Taking advantage of the orthogonality of $SL^{(1)}(2,R)$ subrepresentations, the probability for a state to have a definite *radius*, can be computed by simply using the orthogonal projector on the appropriate $SL^{(1)}(2,R)$ representation.

This is the essence of our quantum-gravity model: the Universe is not just a space-time (a $SL^{(1)}(2,R)$ representation), but the whole ensemble of them. A state of the Universe is a superposition of space-times with different radii (states in different $SL^{(1)}(2,R)$ representations). We cannot speak of the *radius* of the Universe; only the probability that the Universe has a certain *radius* makes real sense. The effect of gravity is that of changing the *radii* distribution of the Universe ($\hat{H}_{n\geq 2}$ move the distribution towards smaller *radii*, while $\hat{H}_{n\leq 2}$ bring about larger *radii*) and, on a specific space-time, producing linear excitations ($\hat{H}_{|n|=1}$) which eventually might be interpreted as quantum states of a free "particle" of mass $m = m(N)$ moving on this AdS space-time [11].

It should be stressed that, since we are dealing with maximal-weight representations, the net effect of the gravitational modes is the decreasing of the average *radius* ($\hat{H}_{n\geq 2}$ eventually annhilate a given state of the Universe, while $\hat{H}_{n\leq 2}$ do not).

A remarkable property of the underlying symmetry, the Virasoro group, is that (as pointed out in the **Introduction**) it can be realized as the diffeomorphism group of a given

[11] A specific determination of m could be made by comparing the present $SL(2,R)$ states with those found in [25, 26, 23]. In fact, the wave functions in those papers (when restricted to the (1+1)-dimensional case) support an irreducible representation of $SL(2,R)$ with index $N = \frac{cR}{m\hbar}$, where in this expression c is the speed of light, $\hbar$ the standard Planck constant and R the *radius* of the AdS space-time.

manifold (S^1). Thus, the quantum operators of the theory can be thought of as being the quantum version of (non-linear or general) changes of reference, traditionally considered as gauge transformations. In the present model, the Virasoro (quantum) operators generate true dynamical changes in the sense that they have a non-trivial action on the Hilbert space (quantum solution manifold). For instance, the operator $\hat{H}_2$ takes the state $\mid N = 4, n = 0\rangle \equiv (-\frac{3}{5}\hat{L}_{-4} + \hat{L}_{-2}\hat{L}_{-2}) \mid 0\rangle$, representing a ground AdS space-time of *radius* $\frac{c}{\sqrt{3\cdot 4}}$ to another ground AdS space-time, but this time of *radius* $\frac{c}{\sqrt{1\cdot 2}}$. Only for $c \to \infty$, the classical limit (see next subsection), this transformation can be considered as a gauge transformation.

In fact, at this limit, we find that the energy of the ground state goes to zero ($Energy \mid N, 0\rangle \to 0$) and the *radius* to infinity ($R = \frac{c}{\sqrt{N(N-1)}} \to \infty$) for all the space-times. Therefore, they are physically indistinguishable and it makes sense to identify them, resulting in the existence of a single space-time in the classical limit $c \to \infty$. This implies the loss of dynamical content of the $\hat{L}_n$ modes, which act as gauge transformations in that single space. We recover the gauge nature of the diffeomorphism but only in the classical limit. The solution manifold under the diffeomorphism constraints would go to a one-degree of freedom phase-space, one q and one p (which is formally equivalent to that of a single particle moving in a fixed space-time). This is a rather standard situation in other approaches to 2D-gravity, where the diffeomorphism constraints are imposed prior to the quantization [27].

3.3 The classical limit

Finally, let us consider the (semi)-classical limit of the model. The main interest of this limit is really the justification of the statements made about the different constants which appeared in the previous subsection.

It can be argued [9], using the Virasoro Poisson brackets (in the original form (3)), that the semiclassical region for the quantization of the Virasoro group corresponds to large values of the true cohomology parameter c. The Planck constant proves to be $\sim \frac{1}{c}$ ($\frac{a}{c}$ when the dimensional constant a is introduced); that is, in the semiclassical region, the fundamental distance c is much larger than a.

Consistency with the classical limit is the reason for chosing c as being related to the large fundamental distance (and eventually to the Universe *radius*) and a to the small one. The condition that characterizes the class of Virasoro representations under study (i.e. $c > 1$) prevents the long distance c from getting smaller than the short length a. In fact, it imposes $\frac{c}{a} > 1$ (the dimensionally correct version of ($c > 1$), so that we always have $c > a$). Long and short fundamental distances are, in this way, well-defined notions in the sense that they do not cross each other. This is no longer valid, however, for the severe quantum region, $c < 1$.

The *radius* is thus $\frac{c}{\sqrt{N(N-1)}}$. Therefore, a semi-classical region of the system (large c) corresponds to a large value of the *radius* R of our space-time support. With respect to the metric on the hyperboloid, this imposes that $\mid K \mid \ll 1$, so that we approach a

Minkowski space-time [12].

Let us develop the classical limit in more detail and then compare it with the classical formalism described at the end of **Section 2**, in order the identify the physical content of the theory at this phenomenological limit.

At the quantum level the dynamics is described by the action of the modes $\hat{L}_n$ ($|n| \geq 2$), the effect of which is that of mixing states of space-times with (in general) different radii, thus affecting a quantum notion of *distance*. In the configuration-space description the dynamics can be encoded in the field operator $\hat{F}$, obtained from the expression of $F(\lambda, u, v)$ in (14) by replacing the constants $\mathcal{L}_n$ (or equivalently L_n's) with the corresponding quantum operators $\hat{L}_n$. At the classical limit, where there is a single space-time (as explained at the end of the previous section), the phenomenological result of the dynamical transformations must be relegated to that of producing changes in the classical *distance*. In a classical theory, the object that models such changes in the *distance* is a dynamical metric field. Thus, in the limit $c \to \infty$ of this theory, we expect the field $F(\lambda, u, v)$ to be associated with the dynamical part of a metric.

More precisely, the metric tensor must adopt the form

$$g^{\mu\nu} = \eta^{\mu\nu} \ + \ g^{\mu\nu}{}_{dyn} \tag{37}$$

where $\eta^{\mu\nu}$ is the background metric inherited from the rigid AdS metric of each of the coexisting space-times in the quantum theory (and it is associated with the kinematical degrees of freedom $L_0, L_{\pm 1}$) and $g^{\mu\nu}{}_{dyn}$ is the dynamical part, which must be derived in terms of the classical field $F(\lambda, u, v)$.

To determine the explicit form of the metric $g^{\mu\nu}{}_{dyn}$, we resort to the classical formulation developed in the last part of **Section 2**.

Firstly, we constrain the $SL(2,R)$ parameters λ, u, v in F, in order to fall down to an AdS space-time of radius R (this is the classical analogue of the restriction in the quantum theory to an $SL(2,R)$ representation by imposing the Casimir constraint). Thus, from the expression obtained in **Section 2**,

$$F(u, v, \lambda) = \lambda + \sum_{n \neq 0} l_n(u, v) e^{in\lambda} \ , \tag{38}$$

we obtain

$$\begin{aligned} F_{AdS}(u, \lambda) \ &\equiv \ \int dv \ \delta(v^2 + \bar{\lambda}^2 - u^2 - R^2) F(u, v, \lambda) = \\ &= \ \int dv \ \delta(v^2 + \bar{\lambda}^2 - u^2 - R^2)(\lambda + \sum_{n \neq 0} l_n(u, v) e^{in\lambda}) \ . \end{aligned}$$

This constraint forces v to be of the form

$$v = \sqrt{R^2 - \hat{\lambda}^2 + u^2} = R\sqrt{1 - (\frac{\bar{\lambda}}{R})^2 + (\frac{u}{R})^2} \ , \tag{39}$$

[12] In loose terms (we repeat that there is no cosmological constant in two dimensions), the "cosmological constant" Λ ($\sim K$) goes to zero in the semi-classical region.

which in the classical limit, $R \to \infty$, simplifies to $v \sim R$. Therefore:

$$\begin{array}{rcl} l_n(u,v) & \sim & l_n(u, R\to\infty) = l_n(u) \\ F_{AdS}^{R\to\infty}(u,\lambda) & = & \lambda + \sum_{n\neq 0} l_n(u) e^{in\lambda} \equiv F(u,\lambda) \end{array} \tag{40}$$

This expression can be directly inverted and the expression for the l_n of reference [4], as Fourier coefficients of the diffeomorphisms of S^1, is recovered. From this, and the explicit form of Θ, we get the expression for the action of the field $F(u,\lambda)$:

$$S = \int \Theta = -\frac{c}{48\pi}\left(\int dud\lambda \frac{\partial_u F}{\partial_\lambda F}\left(\frac{\partial^3_\lambda F}{\partial_\lambda F} - 2\frac{(\partial^2_\lambda F)^2}{(\partial_\lambda F)^2}\right) + \int dud\lambda \frac{\partial_u F}{\partial_\lambda F}(\partial_\lambda F - 1)\right) \ . \tag{41}$$

At this point, we recognize the form of Polyakov's action (1) with a corrective term. The role of the light-cone variables x^- and x^+ is played by λ and u, respectively. Repeating in reverse order the arguments of [1], we identify the previous expression with the action of a dynamical metric of the form,

$$ds^2_{dyn} = \partial_\lambda F d\lambda du \ , \tag{42}$$

which arises as linked to the conformal anomaly [13].

Thus, the complete form of the metric on the space-time at the classical limit is:

$$ds^2 = ds^2_{AdS(R\gg 1)} \ + \ \partial_\lambda F d\lambda du \ . \tag{43}$$

We see that, due to the presence of the background term, the nature of λ and u is no longer that of light-cone variables, but rather of time and space, as dictated from the AdS metric. Only in the regime ($c \to \infty$, $\mid \partial_\lambda F \mid \gg 1$) can the background term be neglected and can we properly recover the corrected Polyakov action. The corrective term has already been found in the literature [28], where it was interpreted as being related to an outer field U, whereas here it is crucial for the consistency of space-time.

4 Conclusions

We have reviewed the Virasoro group as the basic symmetry of a model for two-dimensional quantum gravity (without matter), avoiding the assumption of the existence of external parameters which build the space-time manifold. In this context, we have seen that such space-time emerges only for the critical value of the anomaly $c = c'$, as a consequence of the fundamental role played by cohomology (and pseudo-cohomology) in the determination of the dynamical content of the degrees of freedom of a theory. Nevertheless, a well-defined theory out of this critical value of the extension does exist (we have an

[13] The conformal anomaly is present in our model, as can be computed from the Noether invariants L_n with the Poisson bracket derived from $d\Theta$, or directly from the commutators of the quantum operators $\hat{L}_n$. Note that if $c = 0$, the action vanishes.

explicit realization of the algebra of operators) and we can argue that even in those cases in which the notion of space-time makes no sense, we have a "physical" system which evolves according to some proper time.

If we insist on the notion of space-time, non-commutative geometry ideas are well suited for the implementation of this notion. In fact, non-commutative C^*-algebras leading to non-commutative geometries can occur if higher-order polarizations are needed to reduce the representations. In generalizations of this model one must be prepared to deal with non-commutative geometry.

The notion of space-time in our approach is rather unusual, if compared with other schemes, in some respects here summarized:

- It appears only for a critical value of the central extensions of the group.

- For a given value of $c = c'$, it is a superposition of standard space-times with different *radii* (the different representations of $SL(2, R)$ that appear in the Virasoro representation), with a weight given by the degeneration factors.

- The quantum analogues of general changes of variables are not necessarily gauge transformations. General covariance may be properly realized in the classical limit. It reinforces the idea that diffeomorphism constraints should not be fully imposed prior to quantization.

- We have found a relationship between two fundamental constants, the curvature K (related with the *radius* R of the Universe) and the Planck constant $\hbar$ (related to c [14]; see [9]):

  $$K \sim R^{-2} \sim c^{-2} \sim \hbar^2. \tag{44}$$

  Thus, if we look at the Universe in a classical way ($h \to 0$) we find that the curvature goes to zero.

Although the fundamental goal of the present paper was to clarify the way space-time notion emerges, the introduction of matter in the model should be studied next. This can be accomplished by considering the semi-direct action of Virasoro on a Kac-Moody group. Support for this idea can be found in [29], where, by the use of a completely different approach, the structure of the solution-space manifold for 2D gravity with matter is identified as a $\frac{W \otimes_s G^\infty}{K \otimes_s H^\infty}$ homogeneous space (something expected in the quantization of a Kac-Moody group with a Virasoro semidirect action). The important point is that the present work suggests the separation of the problem of space-time from that of matter in the $W \otimes_s G^\infty$ quantization.

Another unavoidable question is that we have not dealt with Einstenian gravity, but rather with a higher-order correction to it. In two dimensions, classical Einstein gravity is trivial, but in going from $1 + 1$ to $3 + 1$ dimensions, we should find an analogue of

[14]Note that this is an "interaction" Planck constant in the sense that it is essentially related to gravity. Here it is important with respect to the classical limit.

the Virasoro group and, in addition, a precise framework through which Einstenian (or a quantum version of it) enters the scene. Also, going to higher dimensions within the present scheme opens the possibility of having natural transitions between space-times with different topologies as homogeneous spaces associated with a characteristic subgroup larger than $SL(2,R)$.

A final general remark is that in GAQ any generator in the characteristic subalgebra can be written as a function of the dynamical ones (that is, the basic ones). In our model this means that space-time generators are expressed in terms of quantum gravity operators: space-time is thus constructed from interaction.

5 Appendix A

- Reduction of $\mathcal{H}_{(c,c)}$ under $SL^{(1)}(2,R)$.

  i) $SL^{(1)}(2,R)$ maximal-weight vectors have definite a level.

  Proof: let $\mid m\rangle$ be in an irreducible representation of $SL^{(1)}(2,R)$, satisfying $\hat{L}_1 \mid m\rangle = 0$ (maximal-weight vector, which is unique in the representation). Let us consider the vector $\hat{L}_0 \mid m\rangle$, which is in the same representation that $\mid m\rangle$. Then

$$\hat{L}_1(\hat{L}_0 \mid m\rangle) = (\hat{L}_1 + \hat{L}_0\hat{L}_1) \mid m\rangle = 0 \quad . \tag{45}$$

  That is, $\hat{L}_0 \mid m\rangle$ is also a maximal weight vector, which implies:

$$\hat{L}_0 \mid m\rangle = N \mid m\rangle \quad \Rightarrow \mid m\rangle \text{ eigenvector of } L_0. \tag{46}$$

  Furthermore, the value of the Casimir on $\mid m\rangle$ (and in all the representation) is $N(N-1)$:

$$(\hat{L}_0^2 - \frac{1}{2}(\hat{L}_1\hat{L}_{-1} + \hat{L}_{-1}\hat{L}_1)) \mid m\rangle = (\hat{L}_0^2 - \hat{L}_0) \mid m\rangle = (N^2 - N) \mid m\rangle = N(N-1) \mid m\rangle \tag{47}$$

  ii) Vectors inside an irreducible representation have a level higher than the level of their maximal-weight vector.

  Proof: Directly from the construction of the vectors $(\hat{L}_{-1})^n \mid m\rangle$.

  iii) In the level N there are $D^{(N)} - D^{(N-1)}$ maximal-weight vectors, where $D^{(N)}$ is the dimension of the level N.

  Proof: We use induction on N.

  For N=2 ($D^{(2)} - D^{(1)} = 1 - 0 = 1$), and in fact the only independent vector in the $N = 2$ level, $\hat{L}_{-2} \mid 0\rangle$, is a maximal-weight vector: $\hat{L}_1\hat{L}_{-2} \mid 0\rangle = 0$ (we can also check the validity of our assertion for $N = 3$, $D^{(3)} - D^{(2)} = 1 - 1 = 0$, or $N = 4$, $D^{(4)} - D^{(3)} = 2 - 1 = 1$).

Assuming it for $N-1$, let us consider the level N. There are $D^{(N)}$ independent vectors, $D^{(N-1)}$ of which belong to representations induced from level $N-1$ by the action of L_{-1}. Therefore we can find $D^{(N)}-D^{(N-1)}$ independent vectors which do not belong to representations constructed from maximal-weight vectors of a lower level and by *ii)* they can neither be obtained from maximal-weight vectors of higher level. Thus, as belonging to some irreducible representation (at this point we are assuming complete reducibility), they have to be maximal-weight vectors themselves. They generate the $D^{(N)}-D^{(N-1)}$ irreducible representations of $N(N-1)$ Casimir (by *i)*).

(Note: *ii)* and *iii)* can be directly bypassed, noting that the homomorphism $\hat{L}_{-1}$ from level N to level $N-1$ is a surjective one. Thus, we use

$$dim(Im\ L_{-1}) + dim(Ker\ L_{-1}) = dim(Level\ N) \tag{48}$$

to achieve the desired result, without assuming complete reducibility, but attaining it in a constructive way).

Finally we have,

$$\mathcal{H}_{(c,c)} = \bigoplus_{N}(D^{(N)} - D^{(N-1)})R^{(N)} \tag{49}$$

- Orthogonality of the $SL(2,R)$ representations.

  i) Different Virasoro levels are orthogonal.

  Let us consider a vector $\hat{L}_{n_j}...\hat{L}_{n_1} \mid 0\rangle$ on level N and the vector $\hat{L}_{m_j}...\hat{L}_{m_1} \mid 0\rangle$ on level $M<N$. When we construct the scalar product, $\langle 0 \mid \hat{L}_{-m_1}...\hat{L}_{-m_j}\hat{L}_{n_j}...\hat{L}_{n_1} \mid 0\rangle$, we observe that the vector $\hat{L}_{-m_1}...\hat{L}_{-m_j}\hat{L}_{n_j}...\hat{L}_{n_1} \mid 0\rangle$ belongs to $N-M$ level, and can be written as a linear combination of a basis of that level. Each element of the basis annhilate $\langle 0 \mid$ by the polarization conditions.

  ii) States of the same level in different $SL(2,R)$ representations are orthogonal.

  Let us consider two maximal weight states $\mid N_1\rangle$ and $\mid N_2\rangle$, corresponding to different representations of level N_1 and N_2 ($N_1 \leq N_2$), respectively. Now, let us consider the scalar product of two states $(\hat{L}_{-1})^{n_1} \mid N_1\rangle$ and $(\hat{L}_{-1})^{n_2} \mid N_2\rangle$, such that $n_1 + N_1 = n_2 + N_2$:

$$\langle N_1 \mid (\hat{L}_1)^{n_1}(\hat{L}_{-1})^{n_2} \mid N_2\rangle = \langle N_1 \mid (\hat{L}_1)^{n_1-n_2}(\hat{L}_1)^{n_2}(\hat{L}_{-1})^{n_2} \mid N_2\rangle \tag{50}$$

  The operator $(\hat{L}_1)^{n_2}(\hat{L}_{-1})^{n_2}$ can always be written in the form $(...)L_1 + L_0$. The first term directly annihilates the vector $\mid N_2\rangle$, while $\mid N_2\rangle$ is an eigenvector of $\hat{L}_0$. Thus,
  a) For $n_1 - n_2 > 0$, $(\hat{L}_1)^{n_1-n_2}$ annihilates $\mid N_2\rangle$.
  b) For $n_1 = n_2$, we can always choose $\mid N_1\rangle$ orthogonal to $\mid N_2\rangle$.

  We have proven the orthogonality of the two vectors.

6 Appendix B

- Expression for invariant vector fields:

$$
\begin{aligned}
\tilde{X}^R_{l^k} &= \frac{\partial}{\partial l^k} + ikl^{m-k}\frac{\partial}{\partial l^m} + \frac{(ik)^2}{2!} l^n l^{m-n-k}\frac{\partial}{\partial l^m} + ... + \frac{(ik)^j}{j!}\sum_{n_1+...+n_j=m-k} l^{n_1}...l^{n_j}\frac{\partial}{\partial l^m} \\
&+ ... + \frac{c}{24}\{(-i)\,k^2\,(-k)\,l^{-k} + \frac{(-i)^2}{2!}k^2 \sum_{n_1+n_2=-k}\left(n_1^2+n_2^2+n_1 n_2\right) l^{n_1} l^{n_2} + ... + \\
&\frac{(-i)^j}{j!}k^2 \sum_{n_1+...+n_j=-k} P^{(j)}\left(n_1,...,n_j\right) l^{n_1}...l^{n_j} + ...\}\Xi - \\
&\frac{c}{24}\{ikl^{-k} + \frac{(ik)^2}{2!} l^{n_1} l^{-k-n_1} + ... + \frac{(ik)^j}{j!}\sum_{n_1+...+n_j=-k} l^{n_1}...l^{n_j} + ...\}\Xi \\
\tilde{X}^R_{\zeta} &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi\,,
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
\tilde{X}^L_{l^k} &= \frac{\partial}{\partial l^k} + i\,(m-k)\,l^{m-k}\frac{\partial}{\partial l^m} - \frac{c^2}{24}k \\
&\left\{(-i)\,(-k)\,l^{-k} + ... + \frac{(-i)^j}{j}\sum_{n_1+...+n_j=-m} n_1...n_j l^{n_1}...l^{n_j} + ...\right\}\Xi \\
&-\frac{c'}{24} i\,(-k)\,l^{-k}\Xi
\end{aligned} \tag{52}
$$

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References

[1] A.M. Polyakov, Mod. Phys. Lett. **A11**, 893 (1987).

[2] A. Alekseev and S. Shatashvili, Nuc. Phys. **B323**, 719 (1989).

[3] V. Aldaya and J.A. de Azcárraga, J. Math. Phys., **23** 1297 (1982).
V. Aldaya, A. Ramírez and J. Navarro-Salas, Commun. Math. Phys. **121** 541 (1989).
V. Aldaya, R. Loll and J. Navarro-Salas, Phys.Lett. **B225** 340 (1989).

[4] V. Aldaya, J. Navarro-Salas and M. Navarro, Phys.Lett. **B260**, 311 (1991).

[5] V. Aldaya, J. Guerrero and G. Marmo, in *Symmetries in Science X*, Ed. Bruno Gruber, Plenum Press (1998), pag. 1-36.

[6] V. Aldaya, J. Navarro-Salas, J. Bisquert, R. Loll, J. Math. Phys. **33**, 3087 (1992).

[7] V. Aldaya, M. Calixto and M. Navarro, Int. J. Mod. Phys. **A12**, 3609 (1997).

[8] V. Aldaya, J. Navarro-Salas, Commun. Math. Phys. **126**, 575 (1990).
V. Aldaya, J. Navarro-Salas, Commun. Math. Phys. **139**, 433 (1991).

[9] E. Witten, Commun. Math. Phys. **114**, 1 (1988).

[10] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, 1972.

[11] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space time*, Cambridge University Press, 1973.

[12] V.G. Kac, Lecture Notes in Physics **94**, 441 (1979).

[13] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett **52**, 1575 (1984).

[14] B.L. Feigin and D.B. Fuchs, Funct. Anal. Appl. **16**, 114 (1982).

[15] C.B. Thorn, Nucl. Phys. **B 248**, 551 (1984).

[16] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal field theory*, Springer-Verlag, 1997.

[17] C. Itzykson, J.M. Drouffe, *Statistical field theory, Volume 2*, Cambrigde Monographs on Mathematical Physics, 1989.

[18] S. Lang, $SL_2(R)$, Addison-Wesley Publishing Company, 1975.

[19] I.M. Gelfan'd, D.A. Raikov and G.E. Chilov: *Les Anneaux normis commutatifs*s, Gauthier-Villars, Paris, 1964.

[20] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[21] I.M. Gel'fand and M.A.Naimark, Mat. Sb. **12**, 197 (1943)
I.M.Gel'fand and M.A.Naimark, Izv. Akad.Naul SSSR **12**, 445 (1948).

[22] V. Aldaya, M. Calixto, J.M. Cerveró, Commun. Math. Phys. **200**, 325 (1999).

[23] V. Aldaya, J. Bisquert and J. Navarro-Salas, Phys. Lett. **A156**, 381 (1991).
V. Aldaya, J. Bisquert, J. Guerrero and J. Navarro-Salas, Rep. Math. Phys. **37**, 387 (1996).

[24] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[25] C. Fronsdal, Phys. Rev. **D10** 589 (1974).

[26] S.J. Avis, C.J. Isham and D. Storey, Phys. Rev. **D18** 3565 (1978).

[27] A. Mikovic, Phys.Lett.**B291** 19 (1992).
J. Navarro-Salas, M. Navarro and V. Aldaya, Nuc.Phys. **B403** 291 (1993).

[28] E. Nissimov and S. Pacheva, Theor. Math. Phys., **93** 1268 (1992).

[29] B. Julia and H. Nicolai, Nuc. Phys. **B482** 431 (1996).